

A probabilistic analysis of a discrete-time evolution in recombination II. (On partitions)

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Abstract

We study the discrete-time evolution of a transformation on a set of probability measures that is up-dated combining independently the marginals on the atoms of partitions. This model was recently introduced in Baake, Baake and Salamat (Discr. and contin. dynam. syst. **36**, 2016) for continuous-time evolution and generalizes previous ones based upon dyadic partitions. We associate to the discrete-time evolution a natural Markov chain and describe its quasi-stationary behavior retrieving all the results we recently found for dyadic partitions.

Keywords: Partitions; Markov chain; Population genetics; Recombination; geometric decay rate; quasi-stationary distributions.

AMS Subject Classification: 60J10; 92D10.

1 Introduction

Here we study the evolution of the following transformation Ξ acting on the set of probability measures μ on a product measurable space $\prod_{i \in I} A_i$,

$$\Xi[\mu] = \sum_{\delta \in \mathcal{G}} \rho_{\delta} \bigotimes_{J \in \delta} \mu_J.$$

Here \mathcal{G} is a set of partitions of the finite set I , $\rho = (\rho_{\delta} : \delta \in \mathcal{G})$ is a probability vector, μ_J is the marginal of μ on $\prod_{i \in J} A_i$, and $\bigotimes_{J \in \delta} \mu_J$ is the product measure.

This transformation was introduced in [4], but in a continuous-time framework as a generalization of dyadic partitions. The study of the dynamics (Ξ^n) based on dyadic partitions, has served as a model of the genetic composition of population under recombination. Most of the works devoted to this evolution have considered the single cross-over case: $I = \{1, \dots, K\}$ and the dyadic partitions (J, J^c) of the type $J = \{i : i < j\}$, $J^c = \{i : i \geq j\}$. We refer to the introductory sections of references [2], [3], [4], [11] and [10] to have a broad perspective of the study of (Ξ^n) in relation to sequence recombination, as well as a detailed description of the works devoted to this subject since the pioneer work of H. Geiringer [7].

Our main results are Theorems 3.3 and Theorem 4.1 shown in Sections 3 and 4, respectively. In the first one we associate to the evolution (Ξ^n) a natural Markov chain (Y_n) whose transition probabilities starting from the coarsest partition, give the coefficients $(b_n(\delta))$ of the decomposition $\Xi^n[\mu] = \sum_{\delta} b_n(\delta) \otimes_{K \in \delta} \mu_K$ written in terms of the product of the marginal measures on the atoms of partitions δ on I . In our second result, which is the main one of this work, we characterize the quasi-stationary behavior of the chain (Y_n) before attaining the product measure $\otimes_{K \in \mathcal{D}(\mathcal{G})} \mu_K$, being $\mathcal{D}(\mathcal{G})$ the common refinement of the partitions in \mathcal{G} . The quasi-stationary results, and their proofs, are entirely similar to those found in [8] for the dyadic case. The unique additional element is that we must prove relations (14) and (15) in Section 4 that in the dyadic case were straightforward. In [8] it is given a detailed discussion about this kind of results. A main interest in quasi-stationarity is because this gives a very precise information on the deviations of the behavior from the limit measure $\otimes_{K \in \mathcal{D}(\mathcal{G})} \mu_K$, and on the other hand because the Markov chain (Y_n) has not the usual irreducibility conditions, see [5, 6, 9].

2 The recombination transformation

First, let us fix some notation on partitions on finite sets. Let I be a finite set. A partition δ of I is a collection of nonempty sets, pairwise disjoint and covering I . We note $\delta = \{L : L \in \delta\}$ and any of the sets L is called an atom of δ . We note by $\mathbb{D}(I)$ the family of partitions of I .

For $\delta, \delta' \in \mathbb{D}(I)$, δ' is said to be finer than δ or δ is coarser than δ' , we note $\delta \preceq \delta'$, if every atom of δ' is contained in an atom of δ . The finer partition is $\{\{i\} : i \in I\}$, and the coarsest one is $\{I\}$. The common refinement between two partitions $\delta, \delta' \in \mathbb{D}(I)$ is noted by $\delta \vee \delta'$ and its atoms are the nonempty elements of the family of sets $\{K \cap K' : K \in \delta, K' \in \delta'\}$. One has $\delta \preceq \delta'$ if

and only if $\delta \vee \delta' = \delta'$.

Let \mathcal{G} be a family of partitions of I . We will associate to it the following collection of partitions. First define $\mathcal{X}_1(\mathcal{G}) = \mathcal{G}$, and by recursion,

$$\forall n \geq 1 : \quad \mathcal{X}_{n+1}(\mathcal{G}) = \{\mathcal{D} \vee \delta : \mathcal{D} \in \mathcal{G}, \delta \in \mathcal{X}_n(\mathcal{G})\}.$$

Since every $\delta \in \mathcal{X}_n(\mathcal{G})$ satisfies $\mathcal{D} \vee \delta = \delta$ for some element $\mathcal{D} \in \mathcal{G}$, we have $\mathcal{X}_n \subseteq \mathcal{X}_{n+1}$ for all $n \geq 1$. This family of sets stabilizes in a finite number of steps, that is there exists $n_0 \geq 1$ such that $\mathcal{X}_{n_0+k}(\mathcal{G}) = \mathcal{X}_{n_0}(\mathcal{G})$ for all $k \geq 0$. Let

$$\mathcal{X}(\mathcal{G}) = \bigcup_{n \geq 1} \mathcal{X}_n(\mathcal{G}) = \mathcal{X}_{n_0}(\mathcal{G}).$$

By construction the common refinement of the partitions in \mathcal{G} ,

$$\mathcal{D}(\mathcal{G}) = \bigvee_{\mathcal{D} \in \mathcal{G}} \mathcal{D}.$$

is the finest partition in $\mathcal{X}(\mathcal{G})$, that is $\delta \preceq \mathcal{D}(\mathcal{G})$ for all $\delta \in \mathcal{X}(\mathcal{G})$.

Remark 2.1. $\mathcal{D}(\mathcal{G})$ is the unique element in $\mathcal{X}(\mathcal{G})$ that satisfies $\mathcal{D}(\mathcal{G}) \vee \mathcal{D} = \mathcal{D}(\mathcal{G})$ for all $\mathcal{D} \in \mathcal{G}$. Moreover, it also holds $\mathcal{D}(\mathcal{G}) \vee \delta = \mathcal{D}(\mathcal{G})$ for all $\delta \in \mathcal{X}(\mathcal{G})$.

On $\mathcal{X}(\mathcal{G})$ we define the relation

$$\delta \rightarrow \delta' \Leftrightarrow [\exists \mathcal{D} \in \mathcal{G} : \delta' = \delta \vee \mathcal{D}]. \quad (1)$$

So, $\delta \rightarrow \delta'$ implies $\delta' \succeq \delta$. Since for every $\delta \in \mathcal{X}(\mathcal{G}_\rho)$ there exists $\mathcal{D} \in \mathcal{G}$ such that $\delta \vee \mathcal{D} = \delta$, we get

$$\forall \delta \in \mathcal{X}(\mathcal{G}_\rho) : \quad \delta \rightarrow \delta. \quad (2)$$

A path between the elements δ and δ' in \mathcal{G} is a sequence $(\delta_k : k = 0, \dots, r)$ in \mathcal{G} such that $\delta_0 = \delta$, $\delta_r = \delta'$ and $\delta_k \rightarrow \delta_{k+1}$ for $k = 1, \dots, r-1$. For every $\delta \in \mathcal{G} \setminus \{I\}$ there exist a path from $\{I\}$ to δ .

Now, let us introduce a product measurable space and the set of probability measures on it. Let (A_i, \mathcal{B}_i) , $i \in I$, be a finite collection of measurable spaces and let $\prod_{i \in I} A_i$ be a product space endowed with the product σ -field $\otimes_{i \in I} \mathcal{B}_i$. Denote by \mathcal{P}_I the set of probability measures on $\prod_{i \in I} A_i$. Let $J \subseteq I$ and \mathcal{P}_J be the set of probability measures on $\prod_{i \in J} A_i$. The marginal $\mu_J \in \mathcal{P}_J$ of $\mu \in \mathcal{P}_I$ on J is,

$$\forall C \in \otimes_{i \in J} \mathcal{B}_i : \quad \mu_J(C) = \mu(C \times \prod_{i \in J^c} A_i)$$

For $J = I$ we have $\mu_I = \mu$, and we put $\mu_\emptyset \equiv 1$ to get consistency in all the relations where it will appear, in particular in product measures.

For all $J, K \subseteq I$, $J \cap K = \emptyset$, $\mu_J \in \mathcal{P}_J$, $\mu_K \in \mathcal{P}_K$, let $\mu_J \otimes \mu_K$ be the product measure. We have that \otimes is commutative and associative, $\mu_\emptyset = 1$ is the unit element, and \otimes is stable under restriction, that is, for all $J, K, M \subseteq I$ with $J \cap K = \emptyset$ and $M \subseteq J \cup K$,

$$(\mu_J \otimes \mu_K)_M = \mu_{J \cap M} \otimes \mu_{K \cap M}. \quad (3)$$

These are the main properties we require from \otimes .

From now on, we fix $\rho = (\rho_\delta : \delta \in \mathbb{D})$ a probability vector, so $\rho_\delta \geq 0$ for $\delta \in \mathbb{D}$ and $\sum_{\delta \in \mathbb{D}} \rho_\delta = 1$. We denote by $\mathcal{G}_\rho = \{\delta \in \mathbb{D} : \rho_\delta > 0\}$ the support of ρ .

Definition 2.2. Define the following transformation $\Xi : \mathcal{P}_I \rightarrow \mathcal{P}_I$,

$$\Xi[\mu] = \sum_{\mathcal{D} \in \mathcal{G}_\rho} \rho_{\mathcal{D}} \bigotimes_{J \in \mathcal{D}} \mu_J. \quad \square$$

We note

$$D^\rho = \mathcal{D}(\mathcal{G}_\rho) = \bigvee_{\mathcal{D} \in \mathcal{G}_\rho} \mathcal{D}.$$

We claim that

$$\mu = \bigotimes_{L \in D^\rho} \mu_L \text{ is a fixed point for } \Xi : \Xi[\mu] = \mu. \quad (4)$$

In fact, from $\mathcal{D}^\rho = \mathcal{D}^\rho \vee \mathcal{D}$ for all $\mathcal{D} \in \mathcal{G}_\rho$, we get $\mu = \bigotimes_{J \in \mathcal{D}} \mu_J$ for all $\mathcal{D} \in \mathcal{G}_\rho$. So, the claim holds.

3 The Markov chain

When $\rho_{\{I\}} = 1$ we get $\Xi[\mu] = \mu$, so Ξ is the identity transformation. Then, in the sequel we assume

$$\rho_{\{I\}} < 1 \text{ or equivalently } \mathcal{G}_\rho \setminus \{I\} \neq \emptyset.$$

Let us define a Markov chain $(Y_n : n \in \mathbb{N})$ with values on $\mathcal{X}(\mathcal{G}_\rho)$. Its transition matrix $P = (P_{\delta, \delta'} : \delta, \delta' \in \mathcal{X}(\mathcal{G}_\rho))$ is given by

$$P_{\delta, \delta'} = \sum_{\mathcal{D} \in \mathcal{G}_\rho : \delta \vee \mathcal{D} = \delta'} \rho_{\mathcal{D}}.$$

Note that P is stochastic because $\sum_{\delta' \in \mathcal{X}(\mathcal{G}_\rho)} P_{\delta, \delta'} = \sum_{\mathcal{D} \in \mathcal{G}_\rho} \rho_{\mathcal{D}} = 1$. From definition and (1) we get

$$P_{\delta, \delta'} > 0 \Leftrightarrow \delta \rightarrow \delta'.$$

From (2) we have $\delta \rightarrow \delta$, and so

$$\forall \delta \in \mathcal{X}(\mathcal{G}_\rho) : \quad P_{\delta, \delta} > 0 \quad (5)$$

Also note that $P_{\delta, \delta'} > 0$ implies $\delta \preceq \delta'$ and so when the chain (Y_n) leaves an state δ it does never return to it.

Remark 3.1. From Remark 2.1 we have $\mathcal{D}^\rho \vee \mathcal{D} = \mathcal{D}^\rho$ for all $\mathcal{D} \in \mathcal{G}_\rho$ and so $P_{\mathcal{D}^\rho, \mathcal{D}^\rho} = 1$ (which is consistent with (4)). Hence, \mathcal{D}^ρ is an absorbing state for the chain (Y_n) and it is the unique absorbing point for this chain.

Remark 3.2. Since there exists a path $\delta_1 = \{I\} \rightarrow \dots \rightarrow \delta_k = \delta$ for all $\delta \in \mathcal{X}(\mathcal{G}_\rho)$, $\delta \neq \{I\}$, this path has positive probability for the Markov chain.

We claim that $P_{\delta, \delta}$ is strictly increasing with \rightarrow , that is

$$\left[\delta \rightarrow \delta', \delta \neq \delta' \right] \Rightarrow P_{\delta, \delta} < P_{\delta', \delta'}. \quad (6)$$

In fact, every $\mathcal{D} \in \mathcal{G}_\rho$ such that $\delta = \delta \vee \mathcal{D}$ also satisfies $\delta' = \delta' \vee \mathcal{D}$. On the other hand there exists $\mathcal{D}_0 \in \mathcal{G}_\rho$ such that $\delta' = \delta \vee \mathcal{D}_0$, and so it also satisfies $\delta' = \delta' \vee \mathcal{D}_0$. We conclude that $P_{\delta', \delta'} \geq P_{\delta, \delta} + \rho_{\mathcal{D}_0}$, so (6) follows.

We denote by \mathbb{P}_δ the law starting from $Y_0 = \delta$ and by $\mathbb{P} = \mathbb{P}_{\{I\}}$ the law of the chain starting from $Y_0 = \{I\}$.

Theorem 3.3. For all $\mu \in \mathcal{P}_I$ we have

$$\Xi^n[\mu] = \sum_{\delta \in \mathcal{X}(\mathcal{G}_\rho)} b_n(\delta) \bigotimes_{K \in \delta} \mu_K$$

with coefficients:

$$\forall \delta \in \mathcal{X}(\mathcal{G}_\rho) : \quad b_n(\delta) = \mathbb{P}(Y_n = \delta).$$

Proof. Let us prove it by induction. Let $n = 0$. We have $\Xi^0[\mu] = \mu$, so we can take $b_0(\{I\}) = 1 = \mathbb{P}(Y_0 = \{I\})$ and $b_0(\delta) = 0 = \mathbb{P}(Y_0 = \delta)$ for every $\delta \neq \{I\}$, so the statement holds.

Assume the statement is satisfied for n , let us show it for $n + 1$. We have

$$\begin{aligned}
\Xi^{n+1}[\mu] &= \Xi^n[\Xi[\mu]] = \sum_{\delta \in \mathcal{X}(\mathcal{G}_\rho)} b_n(\delta) \bigotimes_{K \in \delta} \Xi[\mu]_K \\
&= \sum_{\delta \in \mathcal{X}(\mathcal{G}_\rho)} b_n(\delta) \bigotimes_{K \in \delta} \left(\sum_{\mathcal{D} \in \mathcal{G}_\rho} \rho_{\mathcal{D}} \bigotimes_{J \in \mathcal{D}} \mu_J \right)_K \\
&= \sum_{\delta \in \mathcal{X}(\mathcal{G}_\rho)} b_n(\delta) \bigotimes_{K \in \delta} \left(\sum_{\mathcal{D} \in \mathcal{G}_\rho} \rho_{\mathcal{D}} \bigotimes_{J \in \mathcal{D}} \mu_{J \cap K} \right) \quad (7) \\
&= \sum_{\delta \in \mathcal{X}(\mathcal{G}_\rho)} \sum_{\mathcal{D} \in \mathcal{G}_\rho} b_n(\delta) \rho_{\mathcal{D}} \left(\bigotimes_{K \in \delta} \bigotimes_{J \in \mathcal{D}} \mu_{J \cap K} \right) \\
&= \sum_{\delta \in \mathcal{X}(\mathcal{G}_\rho)} \sum_{\mathcal{D} \in \mathcal{G}_\rho} b_n(\delta) \rho_{\mathcal{D}} \left(\bigotimes_{J \cap K \in \mathcal{D} \vee \delta} \mu_{J \cap K} \right). \quad (8)
\end{aligned}$$

To state (7) we used (3) and in equality (8) we used $\mu_\emptyset = 1$. Therefore we have the decomposition,

$$\Xi^{n+1}[\mu] = \sum_{\delta' \in \mathcal{X}(\mathcal{G}_\rho)} b_{n+1}(\delta') \bigotimes_{M \in \delta'} \mu_M$$

with

$$b_{n+1}(\delta') = \sum_{\delta \in \mathcal{X}(\mathcal{G}_\rho)} \sum_{\mathcal{D} \in \mathcal{G}_\rho: \mathcal{D} \vee \delta = \delta'} b_n(\delta) \rho_{\mathcal{D}} = \sum_{\delta \in \mathcal{X}(\mathcal{G}_\rho)} b_n(\delta) \left(\sum_{\mathcal{D} \in \mathcal{G}_\rho: \mathcal{D} \vee \delta = \delta'} \rho_{\mathcal{D}} \right).$$

So, by induction we can use that the formula holds for n to get,

$$b_{n+1}(\delta') = \sum_{\delta \in \mathcal{X}(\mathcal{G}_\rho)} b_n(\delta) \left(\sum_{\mathcal{D} \in \mathcal{G}_\rho: \mathcal{D} \vee \delta = \delta'} \rho_{\mathcal{D}} \right) = \sum_{\delta \in \mathcal{X}(\mathcal{G}_\rho)} \mathbb{P}(Y_n = \delta) P_{\delta, \delta'} = \mathbb{P}(Y_{n+1} = \delta').$$

□

Remark 3.4. We can expand Ξ^n in terms of rooted trees with root I and where to each node it is associated an element of $\mathcal{X}(\mathcal{G}_\rho)$, in a similar way as done in [8] for dyadic partitions.

4 Quasi-stationary behavior

Let us define the hitting times,

$$\forall B \subseteq \mathcal{X}(\mathcal{G}_\rho) : \quad \zeta_B = \inf\{n \geq 0 : Y_n \in B\}.$$

For $\delta \in \mathcal{X}(\mathcal{G}_\rho)$ we simply put $\zeta_\delta = \zeta_{\{\delta\}}$. For $\delta = \{I\}$ we have $\mathbb{P}(\zeta_{\{I\}} = 0) = 1$. The random time for attaining \mathcal{D}^ρ is simply noted,

$$\zeta = \zeta_{\mathcal{D}^\rho} = \inf\{n \geq 0 : Y_n = \mathcal{D}^\rho\}.$$

Since \mathcal{D}^ρ is an absorbing point, then $Y_{\zeta+n} = \mathcal{D}^\rho$ for all $n \geq 0$. Now, the variables (Y_n) take values in $\mathcal{X}(\mathcal{G}_\rho)$, so we can define the sequence of random probabilities $(\Xi^n[\mu] = \bigotimes_{K \in Y_n} \mu_K)$. Hence, $\Xi^{\zeta+n}[\mu] = \bigotimes_{L \in \mathcal{D}^\rho} \mu_L$ for $n \geq 0$.

Theorem 4.1. *Assume $\rho_I < 1$. Then,*

$$\mathbb{P}(\zeta < \infty) = 1. \quad (9)$$

Let

$$\Delta = \{\delta \in \mathcal{X}(\mathcal{G}_\rho) : \delta \rightarrow \mathcal{D}^\rho, \delta \neq \mathcal{D}^\rho\}.$$

Define

$$\eta = \max\{P_{\delta,\delta} : \delta \in \Delta\} \text{ and } \mathcal{F} = \{\delta \in \Delta : P_{\delta,\delta} = \eta\}.$$

Then, $\eta \in (0, 1)$ and $\mathbb{P}(\zeta_{\mathcal{F}} < \infty) > 0$. The geometric rate of decay of $\mathbb{P}(\zeta > n)$ satisfies,

$$\lim_{n \rightarrow \infty} \eta^{-n} \mathbb{P}(\zeta > n) = \lim_{n \rightarrow \infty} \eta^{-n} \mathbb{P}(\zeta > n, Y_n \in \mathcal{F}) = \mathbb{E}(\eta^{-\zeta_{\mathcal{F}}}, \zeta_{\mathcal{F}} < \infty) \in (0, \infty). \quad (10)$$

Let

$$\mathcal{X}(\mathcal{G}_\rho)^* = \mathcal{X}(\mathcal{G}_\rho) \setminus \{\mathcal{D}^\rho\} \text{ and } P^* = (P_{\delta,\delta'} : \delta, \delta' \in \mathcal{X}(\mathcal{G}_\rho)^*).$$

The quasi-limiting distribution on $\mathcal{X}(\mathcal{G}_\rho)^*$ is given by,

$$\begin{aligned} \forall \delta \in \mathcal{F} : \quad & \lim_{n \rightarrow \infty} \mathbb{P}(Y_n = \delta \mid \zeta > n) = \frac{\mathbb{E}(\eta^{-\zeta_\delta}, \zeta_\delta < \infty)}{\mathbb{E}(\eta^{-\zeta_{\mathcal{F}}}, \zeta_{\mathcal{F}} < \infty)}, \\ \forall \delta \in \mathcal{X}(\mathcal{G}_\rho)^* \setminus \mathcal{F} : \quad & \lim_{n \rightarrow \infty} \mathbb{P}(Y_n = \delta \mid \zeta > n) = 0. \end{aligned} \quad (11)$$

The following ratio limit relation is satisfied for $\delta \in \mathcal{X}(\mathcal{G}_\rho)^*$,

$$\lim_{n \rightarrow \infty} \frac{\mathbb{P}_\delta(\zeta > n)}{\mathbb{P}(\zeta > n)} = \frac{\mathbb{E}_\delta(\eta^{-\zeta_{\mathcal{F}}}, \zeta_{\mathcal{F}} < \infty)}{\mathbb{E}(\eta^{-\zeta_{\mathcal{F}}}, \zeta_{\mathcal{F}} < \infty)}. \quad (12)$$

Both ratios vanish only when $\mathbb{P}_\delta(\zeta_{\mathcal{F}} < \infty) = 0$. The vector

$$\varphi = (\varphi_\delta : \delta \in \mathcal{X}(\mathcal{G}_\rho)^*) \text{ with } \varphi_\delta = \mathbb{E}_\delta(\eta^{-\zeta_{\mathcal{F}}}, \zeta_{\mathcal{F}} < \infty), \quad (13)$$

is a right eigenvector of P^* with eigenvalue η .

Proof. It is obvious that $\eta > 0$ and from Remark (3.1) we have $\eta < 1$. For $\delta \in \mathcal{F}$ we have that $P_{\delta,\delta} > 0$ (see (5)) and $P_{\delta,\mathcal{D}^\rho} > 0$ because $\delta \in \Delta$. Let us prove that,

$$\forall \delta \in \mathcal{F} : \quad P_{\delta,\delta} + P_{\delta,\mathcal{D}^\rho} = 1. \quad (14)$$

Assume $P_{\delta,\delta'} > 0$ for some δ' different from δ and \mathcal{D}^ρ . So, there exists $\mathcal{D}_0 \in \mathcal{G}_\rho$ such that $\delta \vee \mathcal{D}_0 = \delta'$. Now, for any $\mathcal{D} \in \mathcal{G}_\rho$ such that $\delta \vee \mathcal{D} = \mathcal{D}^\rho$ we also have $\delta' \vee \mathcal{D} = \mathcal{D}^\rho$. We deduce that $\delta' \in \Delta$ and that $P_{\delta',\delta'} \geq P_{\delta,\delta} + \rho_{\mathcal{D}_0}$. Hence, $\eta \geq P_{\delta,\delta} + \rho_{\mathcal{D}_0}$, which contradicts $\delta \in \mathcal{F}$. This shows (14). Note that (14) can be written,

$$\forall \delta \in \mathcal{F}, : \quad \delta \rightarrow \delta' \Leftrightarrow [\delta' = \delta \vee \delta' = \mathcal{D}^\rho].$$

Define,

$$\beta_0 = \max\{P_{\delta,\delta} : \delta \in \mathcal{X}(\mathcal{G}_\rho), \delta \neq \mathcal{D}^\rho, \delta \notin \mathcal{F}\}.$$

Let us prove

$$\beta_0 < \eta. \quad (15)$$

If $\delta \in \Delta \setminus \mathcal{F}$, by definition of \mathcal{F} we get $P_{\delta,\delta} < \eta$. Let $\delta \notin \Delta$. It is easy to see that there exists a path $\delta = \delta_0 \rightarrow \delta_1 \rightarrow \dots \rightarrow \delta_r$ for some $\delta_r \in \Delta$ and with all $(\delta_k : k = 0, \dots, r)$ different among them. From (6), P_{δ_k,δ_k} is strictly increasing with k and so $P_{\delta,\delta} < P_{\delta_r,\delta_r}$. Since $P_{\delta_r,\delta_r} < \eta$, relation (15) follows.

Let us show (9). As already noted, when (Y_n) exits from some state it does never return to it. This fact together with inequality $P_{\delta,\delta} < 1$ for $\delta \neq \mathcal{D}^\rho$, give

$$\forall \delta \in \mathcal{X}(\mathcal{G}_\rho), \delta \neq \mathcal{D}^\rho : \quad \mathbb{P}(\#\{n : Y_n = \delta\} < \infty) = 1.$$

So, since \mathcal{D}^ρ is an absorbing state we get (9): $\mathbb{P}(\zeta < \infty) = \mathbb{P}(\exists n : Y_n = \mathcal{D}^\rho) = 1$.

On the other hand, the existence of paths from $\{I\}$ to \mathcal{F} with positive probability gives $\mathbb{P}(\zeta_{\mathcal{F}} < \infty) > 0$.

Let us now turn to the proof of relations (10), (11) and (12). From (14) we get,

$$\forall \delta^* \in \mathcal{F}, n \geq 0 : \quad \mathbb{P}_{\delta^*}(Y_n = \delta^*) = \eta^n.$$

We have

$$\mathbb{P}(\zeta > n) = \mathbb{P}(\zeta > n, Y_n \notin \mathcal{F}) + \mathbb{P}(\zeta > n, Y_n \in \mathcal{F}). \quad (16)$$

Since there exists paths of positive probability from $\{I\}$ to $\delta \in \mathcal{X}(\mathcal{G}_\rho)$, $\delta \neq \{I\}$, we obtain the existence of $k_0 \geq 1$ such that

$$\forall \delta^* \in \mathcal{F} : \quad \mathbb{P}(\zeta_{\delta^*} \leq k_0) > 0.$$

Define $\alpha(\mathcal{F}) := \min\{\mathbb{P}(\zeta_{\delta^*} \leq k_0) : \delta^* \in \mathcal{F}\}$ which is strictly positive. From the Markov property we get for all $\delta^* \in \mathcal{F}$,

$$\begin{aligned} \mathbb{P}(\zeta > n) &\geq \sum_{j=1}^{k_0} \mathbb{P}(\zeta_{\delta^*} = j, \zeta > n) \geq \sum_{j=1}^{k_0} \mathbb{P}(\zeta_{\delta^*} = j) \mathbb{P}_{\delta^*}(\zeta > n-j) \\ &\geq \sum_{j=1}^{k_0} \mathbb{P}(\zeta_{\delta^*} = j) \mathbb{P}_{\delta^*}(Y_{n-j} = \delta^*) \geq \sum_{j=1}^{k_0} \mathbb{P}(\zeta_{\delta^*} = j) \eta^{n-j} \geq \alpha(\mathcal{F}) \eta^n. \end{aligned} \quad (17)$$

To analyze the first term at the right hand side of equality (16), we will use the following simple result, which is proven in detail in Lemma 5.6 in [8]. We have,

$$\forall \theta > 0 \exists C' = C'(\theta) : \mathbb{P}(\forall j \leq n : Y_j \notin \mathcal{F} \cup \{\mathcal{D}^\rho\}) \leq C'(\beta_0 + \theta)^n. \quad (18)$$

We will always take $\theta > 0$ such that $\beta_0 + \theta < \eta$. Hence, from (17) and (18) we find

$$\mathbb{P}(Y_n \notin \mathcal{F} | \zeta > n) \leq C'' ((\beta_0 + \theta)/\eta)^n \rightarrow 0 \text{ as } n \rightarrow \infty, \quad (19)$$

with $C'' = C'/\alpha(\mathcal{F})$. Therefore,

$$\lim_{n \rightarrow \infty} \mathbb{P}(Y_n \in \mathcal{F} | \zeta > n) = 1. \quad (20)$$

Let us examine the second term at the right hand side of equality (16). For every $\delta^* \in \mathcal{F}$ we have

$$\begin{aligned} \mathbb{P}(\zeta > n, Y_n = \delta^*) &= \sum_{j=1}^n \mathbb{P}(\zeta > n, \zeta_{\delta^*} = j) \\ &= \sum_{j=1}^n \mathbb{P}(\zeta_{\delta^*} = j) \mathbb{P}_{\delta^*}(\zeta > n-j) \\ &= \sum_{j=1}^n \mathbb{P}(\zeta_{\delta^*} = j) \eta^{n-j} = \eta^n \left(\sum_{j=1}^n \eta^{-j} \mathbb{P}(\zeta_{\delta^*} = j) \right). \end{aligned}$$

Since

$$\begin{aligned} \mathbb{P}(\zeta_{\delta^*} = j) &\leq \mathbb{P}(\zeta_{\mathcal{F}} = j) \\ &\leq \mathbb{P}(\forall n \leq j-1 : Y_n \notin \mathcal{F} \cup \{\mathcal{D}^\rho\}) \leq C'(\beta_0 + \theta)^{j-1}, \end{aligned}$$

and $\beta_0 + \epsilon < \eta$, we get $\sum_{j=1}^{\infty} \eta^{-j} \mathbb{P}(\zeta_{\delta^*} = j) < \infty$. Hence,

$$\begin{aligned} \forall \delta^* \in \mathcal{F} : \lim_{n \rightarrow \infty} \eta^{-n} \mathbb{P}(\zeta > n, Y_n = \delta^*) &= \sum_{j=1}^{\infty} \eta^{-j} \mathbb{P}(\zeta_{\delta^*} = j) \\ &= \mathbb{E}(\eta^{-\zeta_{\delta^*}}, \zeta_{\delta^*} < \infty) < \infty. \end{aligned} \quad (21)$$

Now, for $\delta^* \in \mathcal{F}$ we have

$$\zeta_{\delta^*} < \infty \Rightarrow [\forall \delta' \in \mathcal{F} \setminus \{\delta^*\} : \zeta_{\delta'} = \infty \text{ and } \zeta_{\mathcal{F}} = \zeta_{\delta^*}].$$

Then,

$$\{\zeta_{\mathcal{F}} = j\} = \bigcup_{\delta^* \in \mathcal{F}} \{\zeta_{\delta^*} = j\}$$

and the union is disjoint. So, $\eta^{-\zeta_{\mathcal{F}}} \mathbf{1}_{\zeta_{\mathcal{F}} < \infty} = \sum_{\delta^* \in \mathcal{F}} \eta^{-\zeta_{\delta^*}} \mathbf{1}_{\zeta_{\delta^*} < \infty}$. Hence,

$$\mathbb{E}(\eta^{-\zeta_{\mathcal{F}}}, \zeta_{\mathcal{F}} < \infty) = \sum_{\delta^* \in \mathcal{F}} \mathbb{E}(\eta^{-\zeta_{\delta^*}}, \zeta_{\delta^*} < \infty) < \infty.$$

Then, from (21), we deduce

$$\lim_{n \rightarrow \infty} \eta^{-n} \mathbb{P}(\zeta > n, Y_n \in \mathcal{F}) = \mathbb{E}(\eta^{-\zeta_{\mathcal{F}}}, \zeta_{\mathcal{F}} < \infty). \quad (22)$$

Therefore, relations (19), (21) and (22), give (11).

Now, relation (10) is a consequence of relations (20) and (22) because they imply

$$\begin{aligned} \lim_{n \rightarrow \infty} \eta^{-n} \mathbb{P}(\zeta > n) &= \lim_{n \rightarrow \infty} \eta^{-n} \mathbb{P}(\zeta > n, Y_n \in \mathcal{F}) \\ &= \mathbb{E}(\eta^{-\zeta_{\mathcal{F}}}, \zeta_{\mathcal{F}} < \infty) \in (0, \infty). \end{aligned}$$

Let us show (12). First, assume δ is such that $\mathbb{P}_{\delta}(\zeta_{\mathcal{F}} < \infty) > 0$. Since there is a path with positive probability from δ to some nonempty subset of \mathcal{F} , a similar proof as the one showing (10) gives

$$\lim_{n \rightarrow \infty} \eta^{-n} \mathbb{P}_{\delta}(\zeta > n) = \mathbb{E}_{\delta}(\eta^{-\zeta_{\mathcal{F}}}, \zeta_{\mathcal{F}} < \infty) \in (0, \infty),$$

so (12) is satisfied. Now, let $\mathbb{P}_{\delta}(\zeta_{\mathcal{F}} < \infty) = 0$. Then, $\mathbb{E}_{\delta}(\eta^{-\zeta_{\mathcal{F}}}, \zeta_{\mathcal{F}} < \infty) = 0$ and in (12) we have $\mathbb{E}_{\delta}(\eta^{-\zeta_{\mathcal{F}}}, \zeta_{\mathcal{F}} < \infty) / \mathbb{E}(\eta^{-\zeta_{\mathcal{F}}}, \zeta_{\mathcal{F}} < \infty) = 0$. We claim that in this case we also have $\lim_{n \rightarrow \infty} \mathbb{P}_{\delta}(\zeta > n) / \mathbb{P}(\zeta > n) = 0$. In fact $\mathbb{P}_{\delta}(\zeta_{\mathcal{F}} < \infty) = 0$ implies

$$\begin{aligned} (\beta_0 + \theta)^{-n} \mathbb{P}_{\delta}(\zeta > n) &= (\beta_0 + \theta)^{-n} \mathbb{P}_{\delta}(\zeta > n, \zeta_{\mathcal{F}} > n) \\ &= (\beta_0 + \theta)^{-n} \mathbb{P}(\forall j \leq n : Y_j \notin (\mathcal{F} \cup \{\mathcal{D}^{\rho}\})) < \infty. \end{aligned}$$

Since $\lim_{n \rightarrow \infty} \eta^{-n} \mathbb{P}(\zeta > n) > 0$ and $\beta_0 + \theta < \eta$, the claim follows and (12) is shown.

The last statement to be proven is that φ defined in (13) is a right eigenvector of P^* with eigenvalue η . First take $\delta \in \mathcal{F}$. We have $\mathbb{P}_\delta(\zeta_{\mathcal{F}} = 0) = 1$ and so $\mathbb{E}_\delta(\eta^{-\zeta_{\mathcal{F}}}, \zeta_{\mathcal{F}} < \infty) = 1$. From (14) and $P_{\delta,\delta} = \eta$ we get

$$(P^*\varphi)_\delta = \sum_{\delta': \delta' \neq D^\rho, \delta \rightarrow \delta'} P_{\delta,\delta'} \mathbb{E}_{\delta'}(\eta^{-\zeta_{\mathcal{F}}}, \zeta_{\mathcal{F}} < \infty) = \eta = \eta \varphi_\delta.$$

Now let δ be such that $\mathbb{P}_\delta(\zeta_{\mathcal{F}} < \infty) = 0$, so $\varphi_\delta = 0$. Then $P_{\delta,\delta'} > 0$ implies $\mathbb{P}_{\delta'}(\zeta_{\mathcal{F}} < \infty) = 0$ and so $(P^*\varphi)_\delta = 0 = \eta \varphi_\delta$.

Now take $\delta \notin \mathcal{F}$ with $\mathbb{P}_\delta(\zeta_{\mathcal{F}} < \infty) > 0$. From the Markov property we get,

$$\begin{aligned} \varphi_\delta &= \mathbb{E}_\delta(\eta^{-\zeta_{\mathcal{F}}}, \zeta_{\mathcal{F}} < \infty) = \sum_{\delta': \delta' \neq D^\rho, \delta \rightarrow \delta'} \mathbb{E}_\delta(\eta^{-\zeta_{\mathcal{F}}}, \zeta_{\mathcal{F}} < \infty, Y_1 = \delta') \\ &= \sum_{\delta': \delta' \neq D^\rho, \delta \rightarrow \delta'} P_{\delta,\delta'} \eta^{-1} \mathbb{E}_{\delta'}(\eta^{-\zeta_{\mathcal{F}}}, \zeta_{\mathcal{F}} < \infty) = \eta^{-1} (P^*\varphi)_\delta. \end{aligned}$$

Then, the result is shown, which finishes the proof of the theorem. \square

From Theorem 4.1 we will obtain two other results: the description of the Q -process, which in our case is the Markov chain that avoids the singleton $\{\otimes_{L \in \mathcal{D}^\rho} \mu_L\}$, and an explicit class of quasi-stationary distributions, that must be compared with the irreducible case where there is a unique quasi-stationary distribution. The Q -process was introduced in [1] for branching processes, and developments on Q -processes in other contexts that include finite Markov chains, are found in [5].

Corollary 4.2. *For all $\delta_i \in \mathcal{X}(\mathcal{G}_\rho)^*$, $i = 1, \dots, k$, the following limit exists*

$$\lim_{n \rightarrow \infty} \mathbb{P}(Y_i = \delta_i, i = 1, \dots, j \mid \zeta > n)$$

and it vanishes if some δ_i satisfies $\mathbb{P}_{\delta_i}(\zeta_{\mathcal{F}} < \infty) = 0$.

Denote

$$\partial(\zeta_{\mathcal{F}}) = \{\delta \in \mathcal{X}(\mathcal{G}_\rho)^* : \mathbb{P}_\delta(\zeta_{\mathcal{F}} < \infty) > 0\}.$$

Then, the matrix $Q = (Q_{\delta,\delta'} : \delta, \delta' \in \partial(\zeta_{\mathcal{F}}))$ given by

$$Q_{\delta,\delta'} = \eta^{-1} P_{\delta,\delta'} \frac{\mathbb{E}_{\delta'}(\eta^{\zeta_{\mathcal{F}}}, \zeta_{\mathcal{F}} < \infty)}{\mathbb{E}_\delta(\eta^{\zeta_{\mathcal{F}}}, \zeta_{\mathcal{F}} < \infty)},$$

is an stochastic matrix on $\partial(\zeta_{\mathcal{F}})$, and it is satisfied

$$\forall \delta_i \in \partial(\zeta_{\mathcal{F}}), i = 0, \dots, j : \lim_{n \rightarrow \infty} \mathbb{P}_{\delta_0}(Y_i = \delta_i, i = 1, \dots, j \mid \zeta > n) = \prod_{i=0}^{j-1} Q_{\delta_i, \delta_{i+1}}.$$

So, Q is the transition matrix of the Markov chain that never hits $\otimes_{L \in \mathcal{D}^\rho} \mu_L$.

Proof. Let us prove that Q is an stochastic matrix. Let φ be the right eigenvector of P^* with eigenvalue η given in (13). The component φ_δ vanishes when $\mathbb{P}_\delta(\zeta_{\mathcal{F}} < \infty) = 0$. Let $\delta \in \partial(\zeta_{\mathcal{F}})$. We will use that $P_{\delta,\delta'} = 0$ if $\delta \not\rightarrow \delta'$ and that

$$\mathbb{P}_{\delta'}(\zeta_{\mathcal{F}} < \infty) = 0 \text{ implies } \frac{\varphi_{\delta'}}{\varphi_\delta} = \frac{\mathbb{E}_{\delta'}(\eta^{\zeta_{\mathcal{F}}}, \zeta_{\mathcal{F}} < \infty)}{\mathbb{E}_\delta(\eta^{\zeta_{\mathcal{F}}}, \zeta_{\mathcal{F}} < \infty)} = 0.$$

Then, since φ is a right eigenvector with eigenvalue η we get

$$\sum_{\delta' \in \partial(\zeta_{\mathcal{F}})} Q_{\delta,\delta'} = \eta^{-1} \left(\sum_{\delta' \in \partial(\zeta_{\mathcal{F}})} P_{\delta,\delta'} \frac{\varphi_{\delta'}}{\varphi_\delta} \right) = \eta^{-1} \left(\sum_{\delta' \in \mathcal{X}(\mathcal{G}_\rho)^*} P_{\delta,\delta'} \frac{\varphi_{\delta'}}{\varphi_\delta} \right) = 1.$$

From the Markov property we obtain for $n > j$,

$$\mathbb{P}(Y_i = \delta_i, i = 1, \dots, j \mid \zeta > n) = \mathbb{P}(Y_i = \delta_i, i = 1, \dots, j) \frac{\mathbb{P}_{\delta_j}(\zeta > n - j)}{\mathbb{P}(\zeta > n)},$$

Now we use the ratio limit result (12). This limit vanishes if $\mathbb{P}_{\delta_j}(\zeta_{\mathcal{F}} < \infty) = 0$ and it also vanishes when $\mathbb{P}_{\delta_i}(\zeta_{\mathcal{F}} < \infty) = 0$ for some $i < j$ because $P_{\delta_i, \delta_{i+1}} > 0$ implies $\mathbb{P}_{\delta_{i+1}}(\zeta_{\mathcal{F}} < \infty) = 0$. For $\delta_i \in \partial(\zeta_{\mathcal{F}})$ for $i = 0, \dots, j$, we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} \mathbb{P}_{\delta_0}(Y_i = \delta_i, i = 1, \dots, j \mid \zeta > n) \\ &= \lim_{n \rightarrow \infty} \mathbb{P}_{\delta_0}(Y_i = \delta_i, i = 1, \dots, j) \frac{\mathbb{P}_{\delta_j}(\zeta > n - j)}{\mathbb{P}_{\delta_0}(\zeta > n)} \\ &= \mathbb{P}_{\delta_0}(Y_i = \delta_i, i = 1, \dots, j) \frac{\varphi_{\delta_j}}{\varphi_{\delta_0}} \eta^{-j} = \prod_{l=0}^{j-1} \left(\eta^{-1} P_{\delta_l, \delta_{l+1}} \frac{\varphi_{\delta_{l+1}}}{\varphi_{\delta_l}} \right). \end{aligned} \quad (23)$$

In (23) we used $\lim_{n \rightarrow \infty} \mathbb{P}(\zeta > n - j) / \mathbb{P}(\zeta > n) = \eta^{-j}$, which is a consequence of (10). Then the result follows. \square

Remark 4.3. *In the above Q -process all the states \mathcal{F} are absorbing states, that is $Q_{\delta^*, \delta^*} = 1$ for all $\delta^* \in \mathcal{F}$. Hence, once the Q -process attains one of the states in \mathcal{F} it remains in it forever.*

Let $\nu = (\nu_\delta : \delta \in \mathcal{X}(\mathcal{G}_\rho)^*)$ be a probability measure on $\mathcal{X}(\mathcal{G}_\rho)^*$. If necessary, ν will be identified with its extension on $\mathcal{X}(\mathcal{G}_\rho)$ with $\nu_{D^\rho} = 0$. We say that ν is supported by some subset $\tilde{\partial} \subseteq \mathcal{X}(\mathcal{G}_\rho)^*$ if $\nu(\tilde{\partial}) = 1$. We denote by ν' the row vector associated to ν .

Corollary 4.4. *Every probability measure ν on $\mathcal{X}(\mathcal{G}_\rho)^*$ supported on \mathcal{F} satisfies $\nu' P^* = \eta \nu'$ and it is a quasi-stationary distribution, that is it satisfies*

$$\forall n \geq 1, \forall \delta \in \mathcal{X}(\mathcal{G}_\rho)^* : \quad \mathbb{P}_\nu(Y_n = \delta \mid \zeta > n) = \nu_\delta. \quad (24)$$

Proof. With the above notation and by using (14) we get,

$$(\nu' P^*)_\delta = P_{\delta, \delta} \nu_\delta = \eta \nu_\delta,$$

so $\nu' P^* = \eta \nu'$. By iteration we find $\nu' P^{*n} = \eta^n \nu'$. Note that this is equivalent to

$$(\nu' P^{*n})_\delta = \mathbb{P}_\nu(Y_n = \delta) = \mathbb{P}_\nu(\forall j \leq n \ Y_j = \delta) = \eta^n \nu'_\delta.$$

Now

$$\mathbb{P}_\nu(\zeta > n) = \sum_{\delta \in \mathcal{F}} (\nu' P^{*n})_\delta = \eta^n \left(\sum_{\delta \in \mathcal{F}} \nu_\delta \right) = \eta^n.$$

Hence, relation (24) is proven. \square

An analogous results can be stated for positive eigenvectors. Let $\tilde{\delta} \subseteq \mathcal{F}$ be a nonempty set, then the characteristic function $\mathbf{1}_{\tilde{\delta}}$ is a right eigenvector of P^* with eigenvalue η .

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